Linear Inequalities and Linear Programming

Systems of Linear Inequalities in Two Variables

- GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES
- SOLVING SYSTEMS OF LINEAR INEQUALITIES GRAPHICALLY
- APPLICATIONS

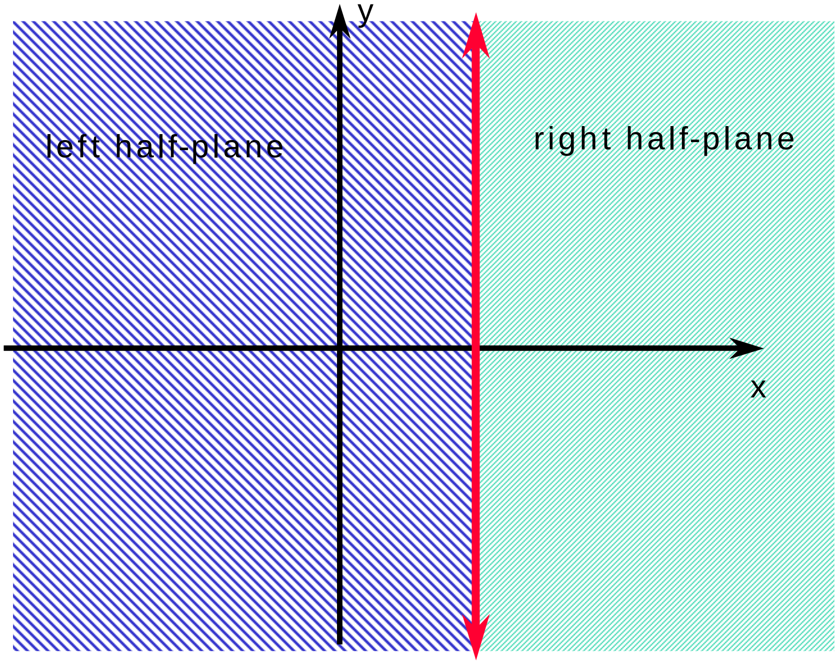
GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

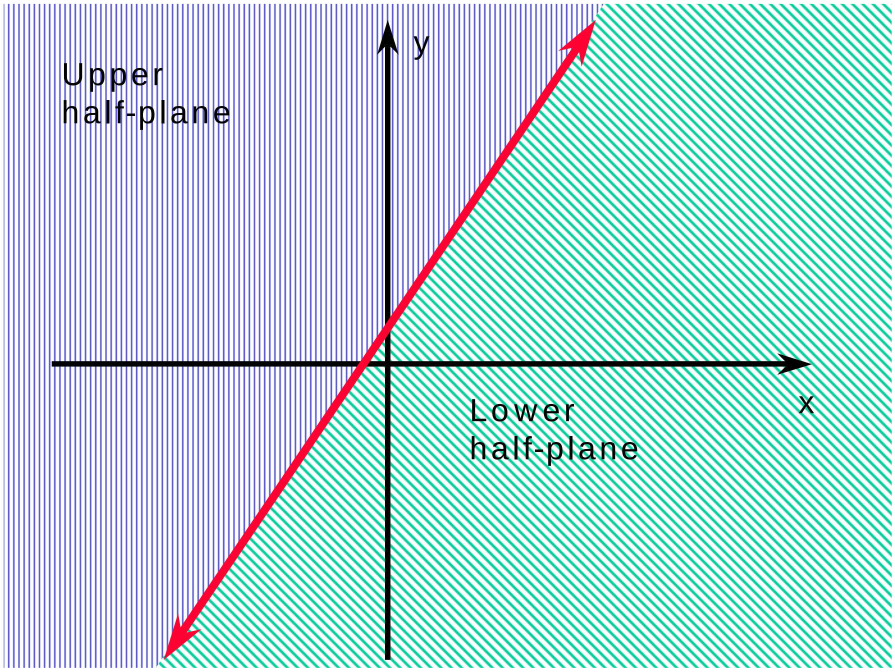
We know how to graph equations

$y=2x-2$ $\qquad$ $2x-2y=4$

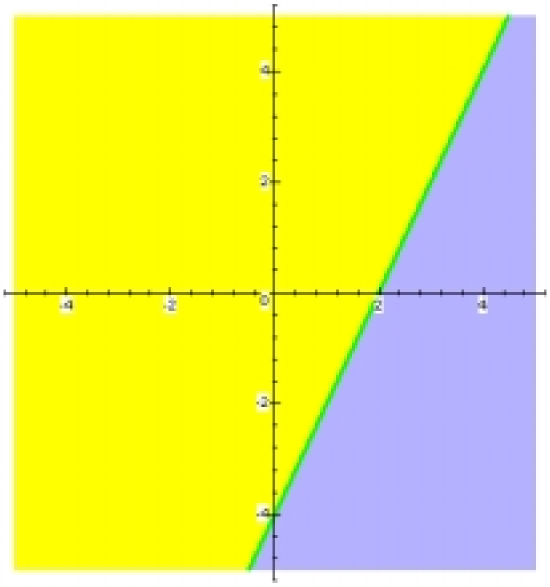
But what about

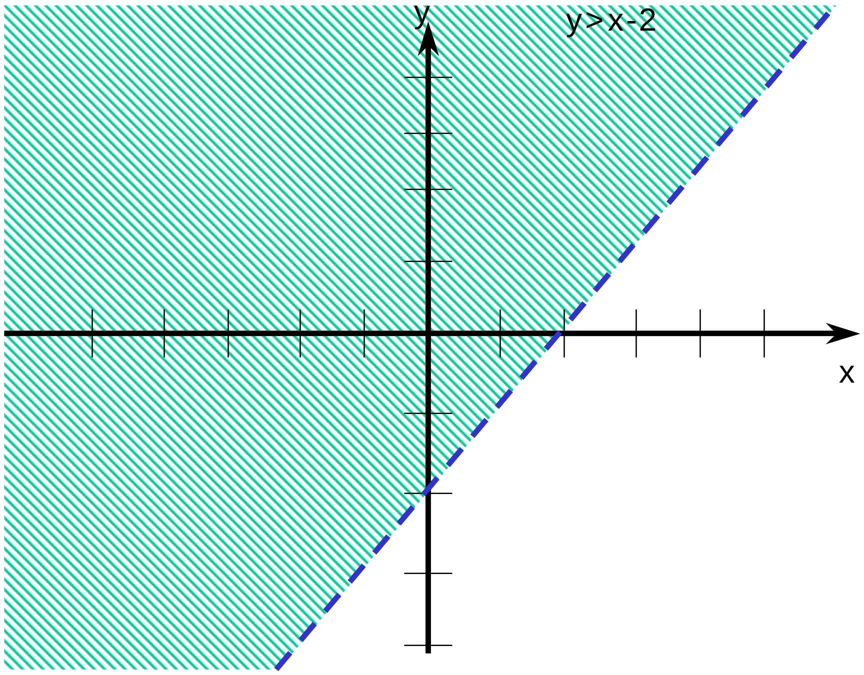
$y\leq 2x-2$ $\qquad$ $2x-2y>4$?

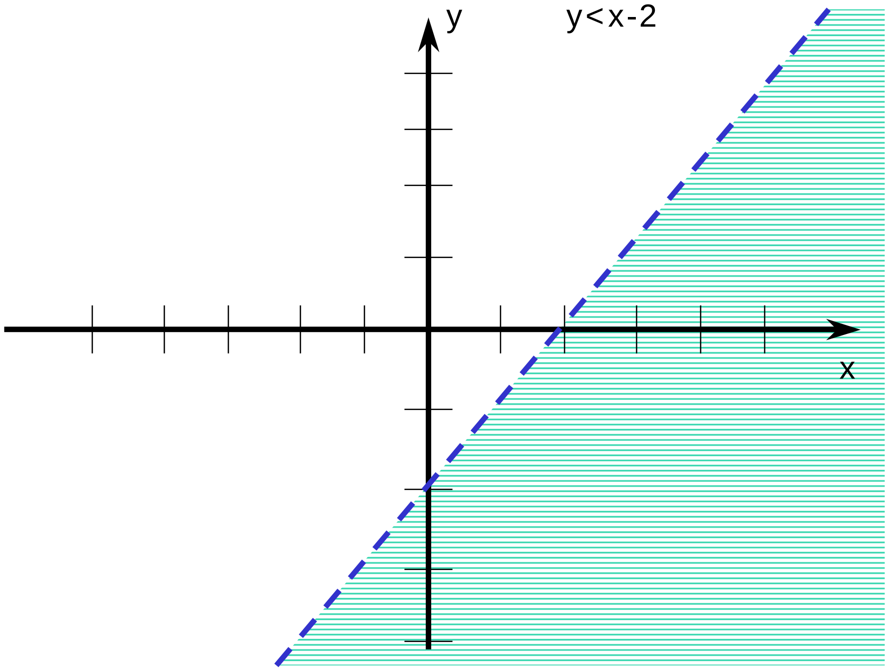


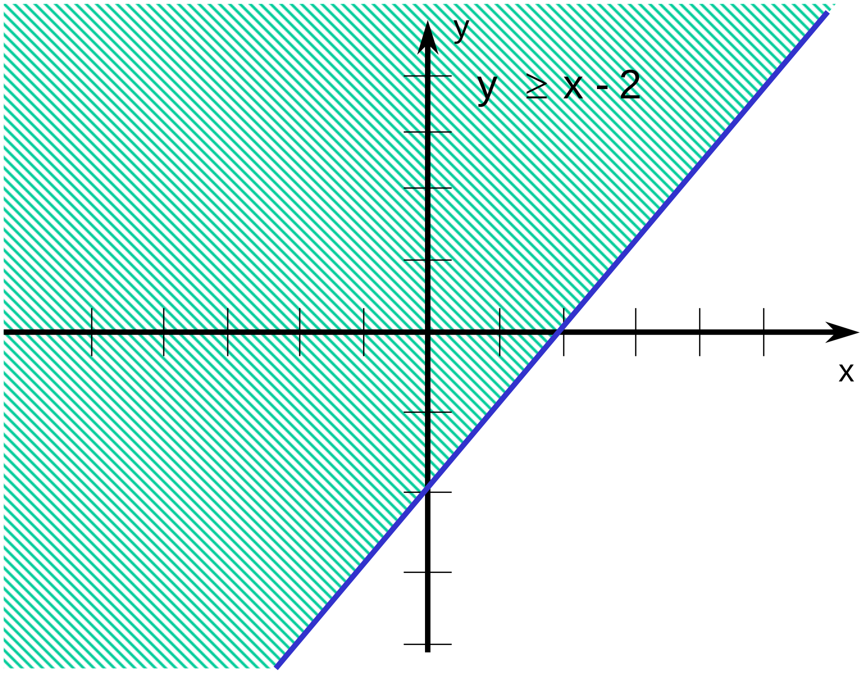


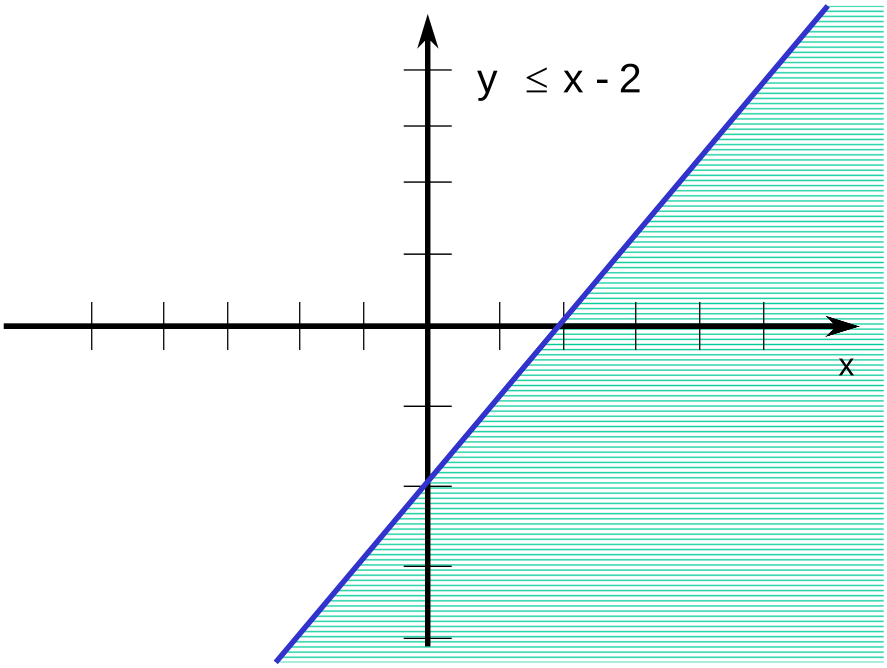
$y \leq 2x-4$




y
y>x-2
x


y
y<x-2
x


y
y ≥ x - 2
x


y ≤ x - 2
x

theorem 1

Graphs of Linear Inequalities

The graph of the linear inequality

$Ax+By<C$ or $Ax+By>C$

with $B \neq 0$, is either the upper half-plane or the lower half-plane (but not both) determined by the line $Ax+By=C$.

If $B=0$, the graph of

$Ax<C$ or $Ax>c$

is either the left half-plane or the right half-plane (but not both) determined by the line

$Ax=C$

Procedure for graphing linear inequalities

Step 1: Graph $Ax+By=C$, dashed if < or >

Step 2: Choose test point not on line [(0,0) is best]
substitute coordinates into inequality

Step 3: If test point coordinates satisfy inequality
shade the half-plane containing it

Otherwise shade other half-plane

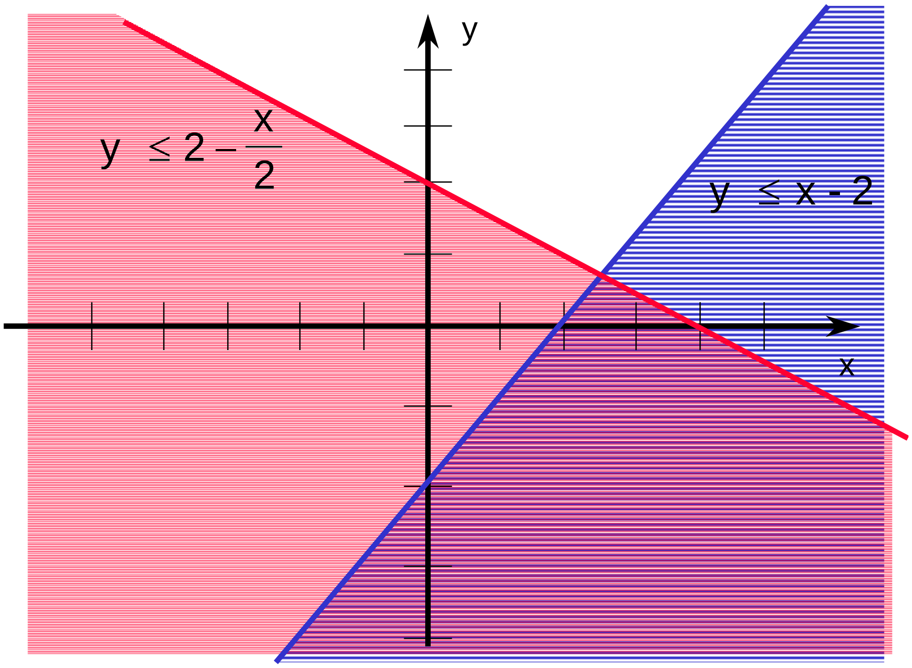
Solving Systems of Linear Inequalities

Step 1: Draw all the lines (dashed if > or <)

Step 2: Shade all regions

Step 3: If all regions overlap then this is the FEASIBLE REGION,

dashed lines are NOT included



Corner Point

A **corner point** of a solution region is a point in the solution region that is the intersection of two boundary lines.

Bounded and Unbounded Solution Regions

A solution region of a system of linear inequalities is **bounded** if it can be enclosed within a circle.

Otherwise it is **unbounded**.

50 Table and Chair production

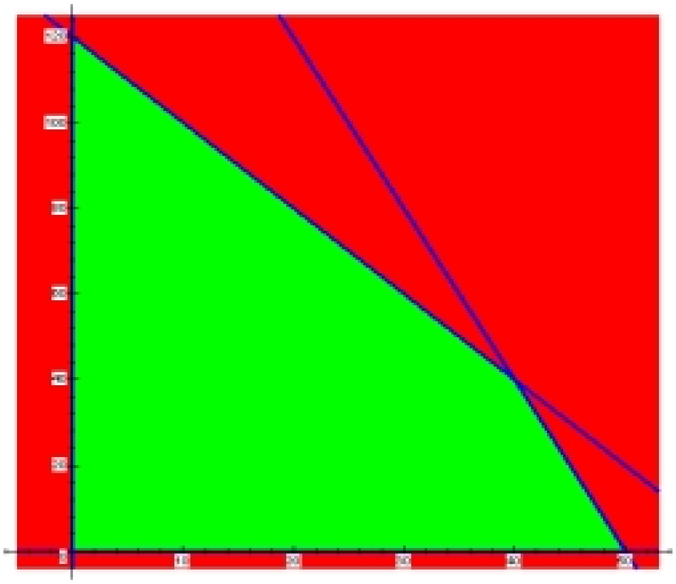
Each table:	8 hours assembly 2 hours finishing
Each chair:	2 hours assembly 1 hour finishing
Hours available:	400 for assembly 120 for finishing

How many tables and chairs can be produced?

Let x be number of tables and y be the number of chairs produced per day.

Assembly:	table 8x, chair 2y
Finishing:	table 2x, chair y

$$\begin{aligned} 8x + 2y &\le 400 \\ 2x + y &\le 120 \\ 0 &\le x \\ 0 &\le y \end{aligned}$$



A) Now suppose each table sells for $50, and each chair for $ 20. How many tables and chairs should we produce to maximize revenue?

B) Suppose that each assembly hour costs $10, while each finishing hour costs $12. How many chairs and tables should we produce to minimize cost?

C) Both A) and B) together, maximize profit.

5.2 Linear Programming in Two Dimensions- A Geometric Approach

- A Linear Programming Problem
- A General Description
- Geometric Solution
- Applications

A Linear Programming Problem

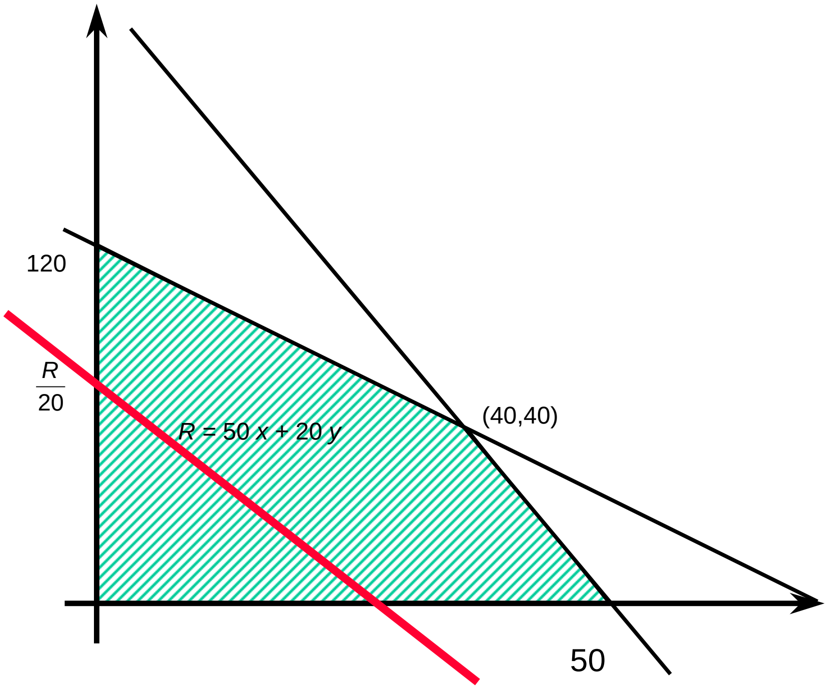
Each table: 8 Hours assembly
2 hours finishing
Each chair: 2 hours assembly
1 hour finishing
Hours available: 400 in the assembly department
120 in the finishing department

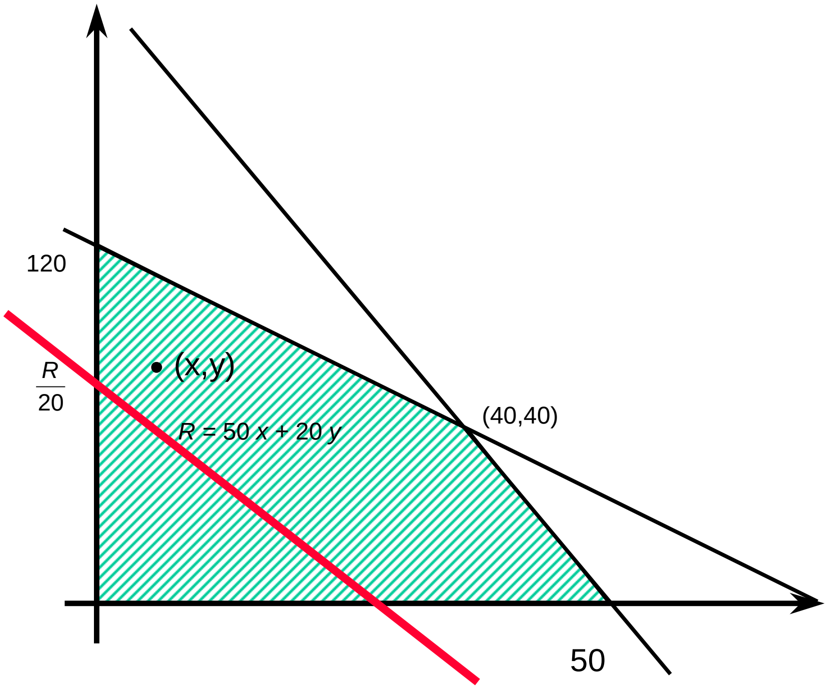
Now suppose each table sells for $ 50 and each chair for $ 20. How many tables and chairs should we produce to maximize revenue?

$$\left.\begin{array}{l} x \text{ number of tables} \\ y \text{ number of chairs} \end{array}\right\} \quad \text{decision variables}$$

$$\text{Constraints} \quad \left\{\begin{array}{l} 8x+2y \le 400 \\ 2x+\ y \le 120 \\ \left.\begin{array}{r} 0 \le x \\ 0 \le y \end{array}\right\} \quad \begin{array}{l} \text{nonnegative} \\ \text{constraints} \end{array} \end{array}\right.$$

objective function $R = 50\,x + 20\,y$



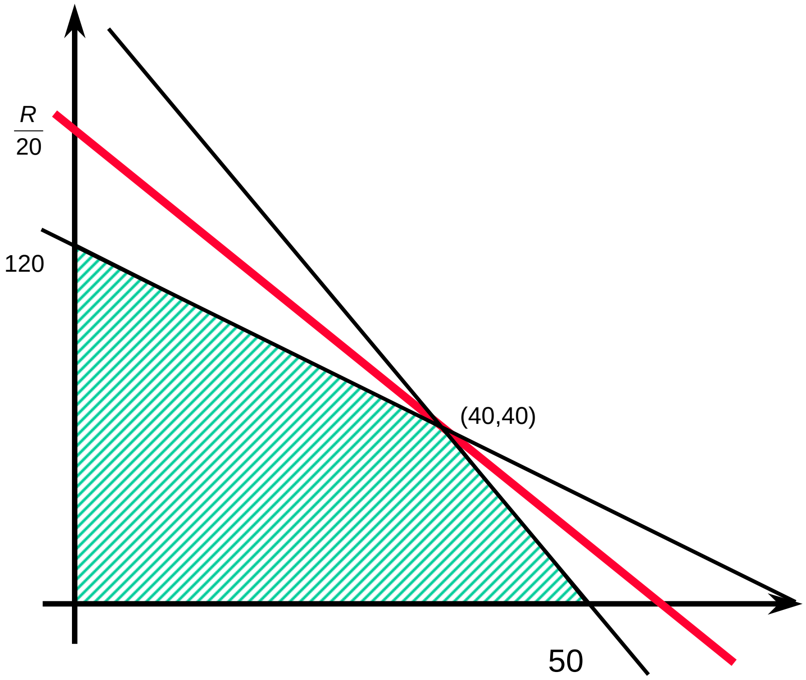


The "higher" the line $R = 50x + 20y$ is, the higher the revenue.

IDEA: move the line as "high" as possible but so that it still intersects the feasible region.

That ought to be the best we can do.

AND IT IS!!!!



So producing 40 tables and 40 chairs should give us a maximum revenue of

$$R = 50 * 40 + 20 * 40 = 2800$$

that is \$ 2800 is the maximum possible revenue, which is attained when we produce 40 tables and 40 chairs.

This shows that corner points are the “extremes” for the objective function, since we use all the resources (available hours in each department), we max out on the constraints.

theorem 1 (version 1)

Fundamental Theorem of Linear Programming

If the optimal value of the objective function exists, then it must occur at one (or more) of the corner points of the feasible region.

theorem 2

Existence of Solutions

(A) If the feasible region is bounded, both max and min of the objective function exist

(B) If the feasible region is unbounded, and the coefficients of the objective function are positive then the min exists

(C) If the feasible region is empty, neither max nor min exist

Graphical Solution

(S1) Summarize information (word problem)

(S2) Form a mathematical model

- (A) Introduce decision variables, write linear objective function
- (B) Write constraints (linear inequalities)
- (C) Write nonnegative constraints

(S3) Graph feasible region, find corner points

(S4) Make table of values of objective function at corner points

(S5) Determine optimal solution from (S4)

(S6) Interpret solution (word problem)

5.3 A Geometric Introduction to the Simplex Method

- STANDARD MAXIMIZATION PROBLEMS
- SLACK VARIABLES
- BASIC AND NONBASIC VARIABLES
- BASIC FEASIBLE SOLUTIONS AND THE SIMPLEX METHOD

STANDARD PROBLEM
STANDARD FORM

Maximize $P = 50x_1+80x_2$ *Objective function*

Subject to $x_1+2x_2 \le \mathbf{32}$ *constraints*
$\mathbf{3}x_1+4x_2 \le \mathbf{84}$

$x_1,x_2 \ge 0$ *nonnegative constraints*

Standard Maximization Problem in Standard Form

A linear programming problem is said to be a ***standard maximization problem in standard form*** if its mathematical model is of the following form:

Maximize the objective function

$$P=c_1x_1+c_2x_2+\ldots+c_nx_n$$

Subject to problem constraints of the form

$$a_1x_1+a_2x_2+\ldots+a_nx_n \le b, \text{ with } b \ge 0$$

and with nonnegative constraints

$$x_1,x_2,x_3,\ldots,x_n \ge 0$$

From Inequalities to Equalities

We know how to deal with equalities, so let us “convert” the inequalities:

$x_1+2x_2 \leq \mathbf{32}$

taking only x_1+2x_2 means there may be some ***slack*** left

So we introduce a *slack variable* $\mathbf{s_1}$**:**

$x_1+2x_2 + s_1 =32$ with $s_1 \geq 0$

Slack variables

Constraints become

$$\begin{aligned} x_1+2x_2 +s_1 \quad\quad &= \mathbf{32} \\ \mathbf{3}x_1+4x_2 \quad\quad + s_2 &= \mathbf{84} \end{aligned}$$

with $x_1=20$ and $x_2=5$ we need $s_1=2$ and $s_2=4$

Problem:

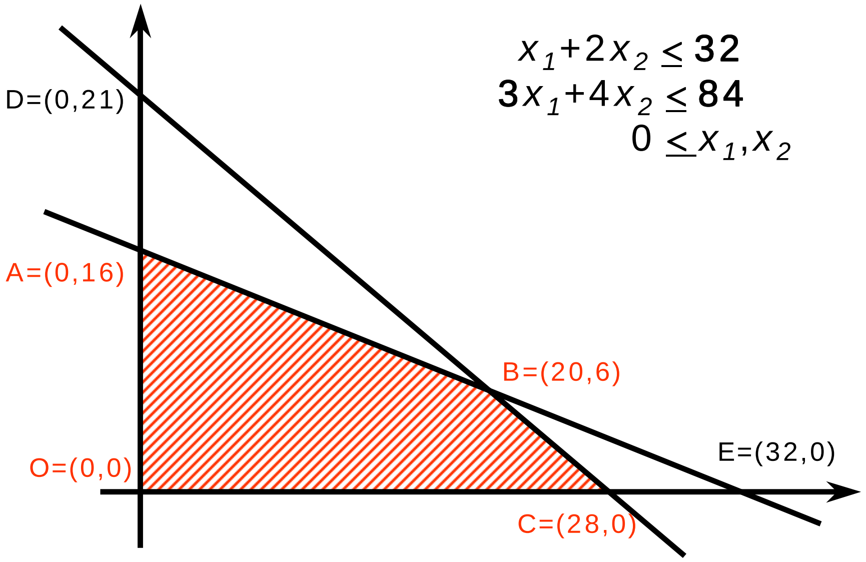
System above has infinitely many solutions!

Basic and Nonbasic Variables
Basic and Basic Feasible Solutions

Corner points are important!

Basic solution correspond to the points of intersection of the boundary lines.

Basic feasible solutions are those basic solutions which lie in the feasible area.



$$x_1+2x_2 \leq \mathbf{32}$$
$$\mathbf{3}x_1+4x_2 \leq \mathbf{84}$$
$$0 \leq x_1, x_2$$

There are 6 basic solutions

A, B, C, D, E, O

But only 4 basic feasible solutions

A, B, C, O

How to find basic solutions?

Select basic variables (free choice), need as many as there are equations.

All other variables are nonbasic.

Now set all nonbasic variables equal to zero and solve for the basic variables.

$$x_1 + 2x_2 + s_1 = 32$$
$$3x_1 + 4x_2 + s_2 = 84$$

2 equations 2 basic
Other two variables are nonbasic.

Since we start with four variables, there are
6 choices for the two basic variables.

(the number of ways we can choose 2 out of 4)

Basic	x_1,x_2	x_1,s_2	s_1,x_2	x_1,s_1	s_2,x_2	s_1,s_2
Nonbasic	s_1,s_2	s_1,x_2	x_1,s_2	s_2,x_2	x_1, s_1	x_1,x_2

Basic or basic feasible solution?

For each choice of basic variables we let the nonbasic variables equal 0, solve for basic.

If there are NO NEGATIVE values for the basic variables then we have a

basic feasible solution.

The final solution

- Determine all basic feasible solutions
- Evaluate the objective function at all the basic feasible solutions
- Pick the one that gives the optimal value

`theorem 1` Fundamental Theorem of Linear Programming -- Version 2

If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the basic feasible solutions.

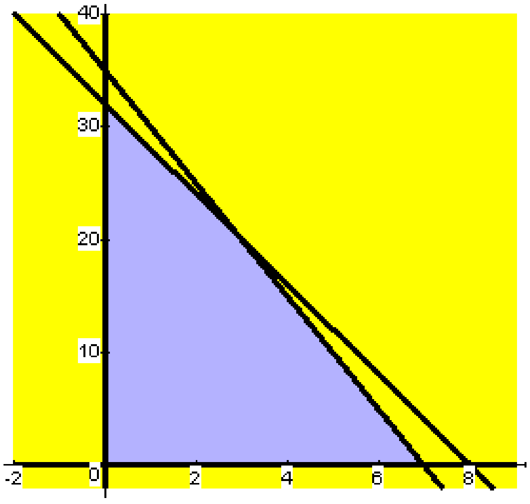
#10 p.301

$5x_1+x_2 \leq 35$
$4x_1+x_2 \leq 32 \qquad 0 \leq x_1,x_2$

$5x_1+x_2+s_1 \quad = 35$
$4x_1+x_2 \quad +s_2 = 32$

2 equations = 2 basic var's

	X_1	*X_2*	*s_1*	*s_2*	*feasible*
(A)	0	0	35	32	YES
(B)	0	35	0	-3	NO
(C)	0	32	3	0	YES
(D)	7	0	0	4	YES
(E)	8	0	-5	0	NO
(F)	3	20	0	0	YES



5.4 The Simplex Method: Maximization with $\le$ constraints

- INITIAL SYSTEM
- THE SIMPLEX TABLEAU
- THE PIVOT OPERATION
- INTERPRETING THE SIMPLEX PROCESS GEOMETRICALLY
- THE SIMPLEX METHOD SUMMARIZED
- APPLICATION

INITIAL SYSTEM

MAXIMIZE: $P=50x_1+80x_2$
SUBJECT TO: $x_1+2x_2\le 32$
$3x_1+4x_2\le 84$
$0\le x_1,x_2$

With slack variables:

$$\begin{array}{rcl} x_1+2x_2+s_1 & & =32 \\ 3x_1+4x_2 & +s_2 & =84 \\ -50x_1-80x_2 & +P & =0 \\ & x_1,x_2,s_1,s_2 & \ge 0 \end{array}$$

Initial System has 5 variables

Basic & Basic Feasible Solutions

1) P is always basic
2) A basic solution of the initial system is also still a basic solution if P is deleted
3) If a basic solution of the initial system is a basic feasible solution after deleting P, then it is a basic feasible solution of the initial system
4) A basic feasible solution of the initial system can contain a negative number but only for P

theorem 1

Fundamental Theorem of Linear Programming--Ver. 3

If the optimal value of the objective function in a linear programming problem exists, then the value must occur at one (or more) of the basic feasible solutions of the initial system.

The Simplex Tableau

$$\begin{array}{cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
s_1 & 1 & 2 & 1 & 0 & 0 & 32 \\
s_2 & 3 & 4 & 0 & 1 & 0 & 84 \\
\hline
P & -50 & -80 & 0 & 0 & 1 & 0
\end{array}$$

Initial Simplex Tableau

Selecting Basic and Nonbasic Variables

Step 1: Determine number of basic and nonbasic variables.

Step 2: Selecting basic variables: Column with exactly one nonzero element, which is not in same row as nonzero element of other basic variable.

Step 3: Selecting nonbasic variables: After all basic variables are selected, the rest are nonbasic

$$\begin{array}{c|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & P & \\
s_1 & 1 & 2 & 1 & 0 & 0 & 32 \\
s_2 & 3 & 4 & 0 & 1 & 0 & 84 \\
\hline
P & -50 & -80 & 0 & 0 & 1 & 0
\end{array}$$

3 equations $\xrightarrow{\text{implies}}$ 3 basic variables

s_1, s_2, P basic
x_1, x_2, nonbasic

Question: How to proceed?

The Pivot Operation

Which nonbasic variable should become basic?

Idea: We want to increase P, choose the one which has the most effect!

$P=50x_1+80x_2$

1 unit increase in x_1 gives $50 more in P
1 unit increase in x_2 gives $80 more in P

Choose x_2!

$$\begin{array}{c|ccccc|c|}
 & x_1 & x_2 & s_1 & s_2 & P & \\
s_1 & 1 & 2 & 1 & 0 & 0 & 32 \\
s_2 & 3 & 4 & 0 & 1 & 0 & 84 \\
\hline
P & -50 & -80 & 0 & 0 & 1 & 0
\end{array}$$

Choose the column that has the
MOST NEGATIVE ENTRY
in the bottom row.

x_2 will enter the set of basic variables
x_2 = entering variable

Column corresponding to entering variable
is the PIVOT COLUMN

Entries in bottom row are INDICATORS

Which variable will become nonbasic (exiting)?
P is always basic, so s_1 or s_2?

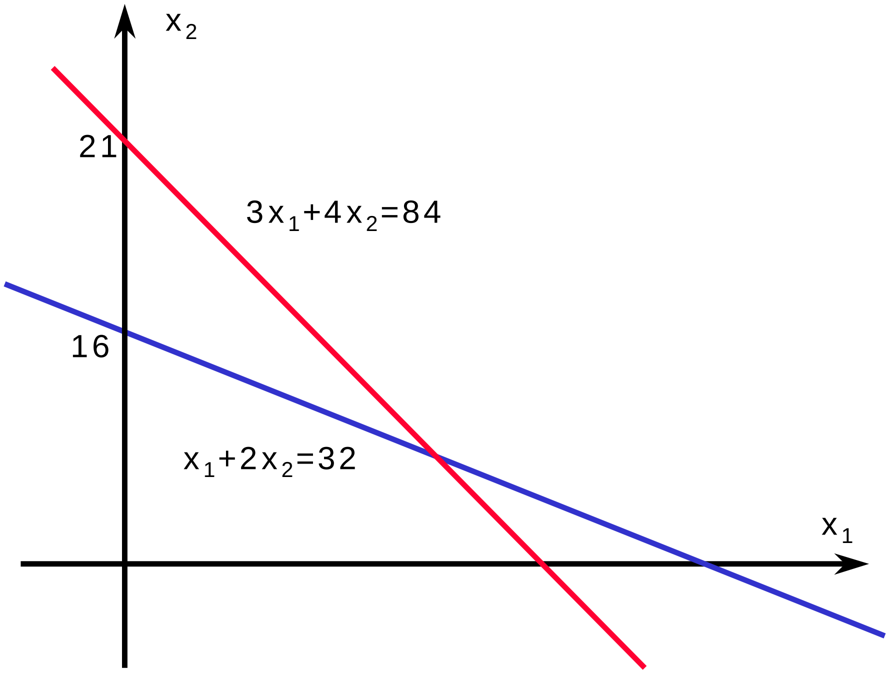
Choosing the exiting variable

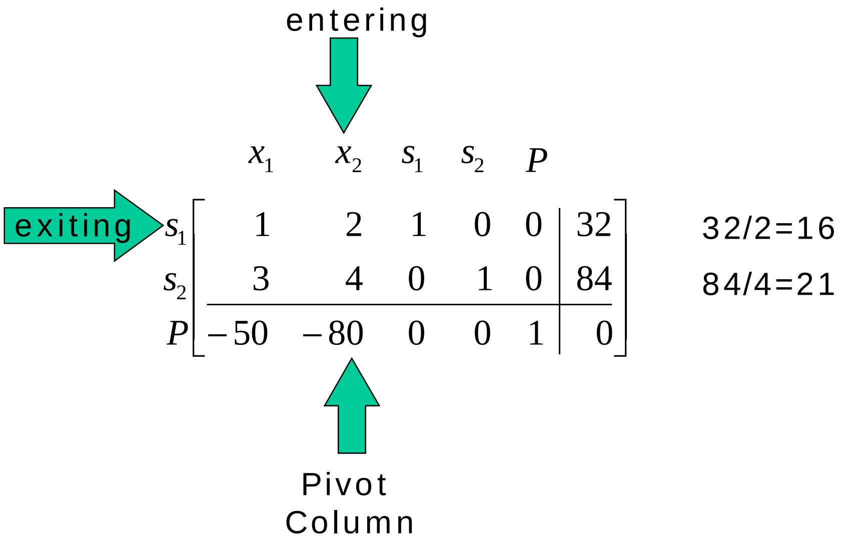
How much can we increase x_2 within the given constraints? Assume x_1=0.

$2x_2+s_1=32$ $\quad$ $s_1=32 - 2x_2 \geq 0$
$4x_2+s_2=84$ $\quad$ $s_2=84 - 4x_2 \geq 0$

$x_2 \leq 32/2=16$
$x_2 \leq 84/4=21$

Both inequalities need to be true, so we need to pick the smaller one. This corresponds to row 1 or s_1.





entering

exiting

$$\begin{array}{c|ccccc|c|} & x_1 & x_2 & s_1 & s_2 & P & \\ s_1 & 1 & 2 & 1 & 0 & 0 & 32 \\ s_2 & 3 & 4 & 0 & 1 & 0 & 84 \\ \hline P & -50 & -80 & 0 & 0 & 1 & 0 \end{array}$$

32/2=16

84/4=21

Pivot Column

Selecting the Pivot Element

Step1: Find most negative element in bottom row
If tie, choose either.

Step2: Divide each positive (>0) element on pivot column into corresponding element of last column. Smallest quotient wins.

Step3: The PIVOT is the element in the pivot column which is in the winning row.

Performing a Pivot Operation

Step 1: Make pivot element 1

Step 2: Create 0's above and below the pivot

We do this using the following elementary row operations:

- multiply a row by a nonzero constant
- add a multiple of one row to another

Effect of Pivot Operation

1. nonbasic variable becomes basic
2. basic variable becomes nonbasic
3. value of objective function increases (or stays the same)

When are we finished?

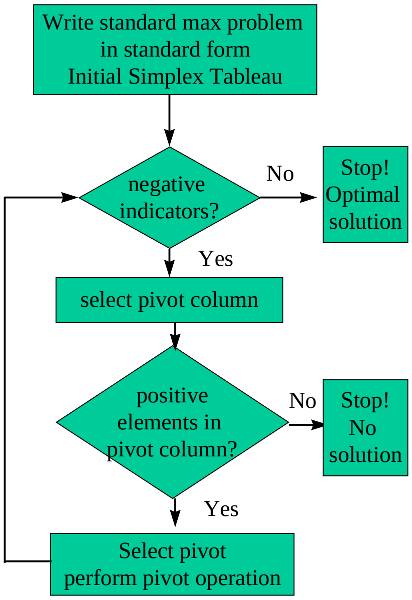
When we run out of things to do!

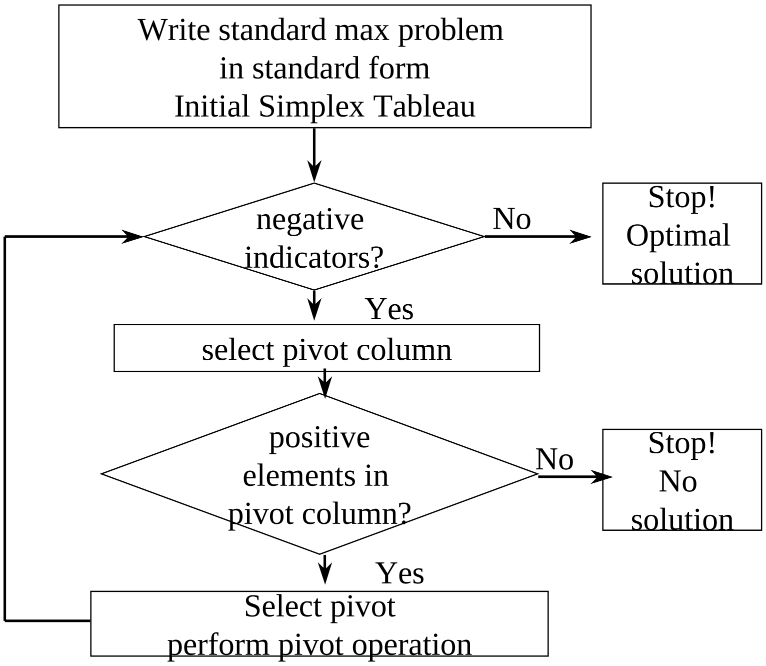
In other words if we can no longer find
a new pivot element.

all indicators are nonnegative

OR

no positive elements in pivot column





5.5 The Dual Problem
Minimization with $\geq$ Constraints

- FORMATION OF THE DUAL PROBLEM
- SOLUTION OF THE MINIMIZATION PROBLEM
- APPLICATION: TRANSPORTATION
- SUMMARY

Formation of the Dual Problem

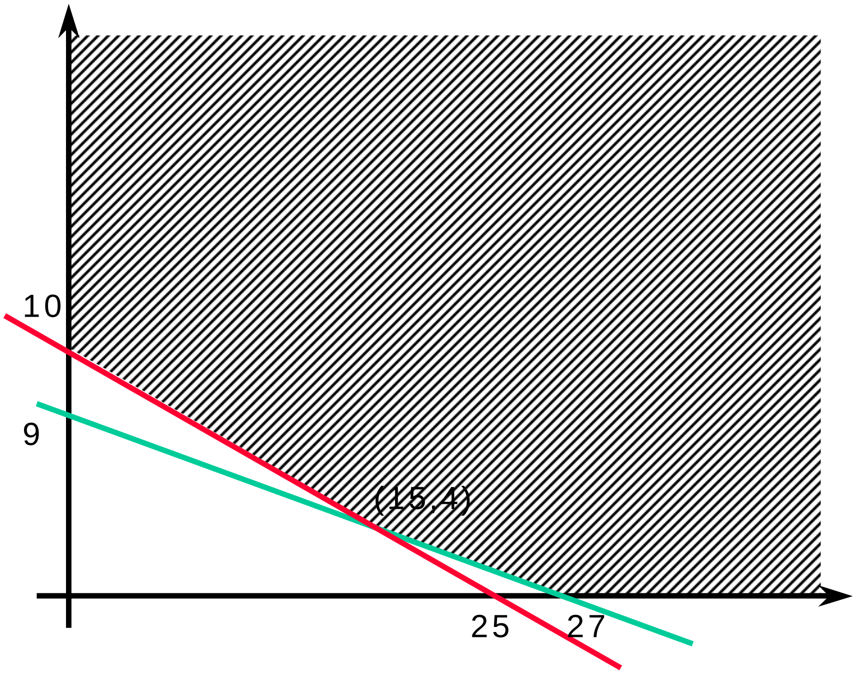
Instead of maximizing profit we might want to minimize cost.

Minimize $C=16x_1+45x_2$

Subject to $2x_1+5x_2 \geq 50$

$x_1+3x_2 \geq 27$

$x_1,x_2 \geq 0$



$$\begin{aligned} 2x_1 + 5x_2 &\geq 50 \\ x_1 + 3x_2 &\geq 27 \\ 16x_1 + 45x_2 &= C \end{aligned}$$

$$A = \left[\begin{array}{cc|c} 2 & 5 & 50 \\ 1 & 3 & 27 \\ \hline 16 & 45 & 1 \end{array}\right]$$

$$A^T = \left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right]$$

Transpose of a Matrix

Given a $(m \times n)$ matrix A, the transpose of A is the $(n \times m)$ matrix A^T obtained by putting the first row of A into the first column of A^T, the second row of A into the second column of A^T, etc.

ROWS of A become COLUMNS of A^T

COLUMNS of A become ROWS of A^T

The Dual Problem

$$A^T = \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right] \end{array}$$

$$\begin{aligned} 2y_1 + y_2 &\leq 16 \\ 5y_1 + 3y_2 &\leq 45 \\ 50y_1 + 27y_2 &= P \end{aligned}$$

MAXIMIZE P subject to $\leq$ constraints

Formation of the Dual Problem

Given a minimization problem with $\geq$ constraints

Step 1: Form matrix A using coefficients of the constraints and the objective function

Step 2: Interchange rows and columns of A to get A^T

Step 3: Use rows of A^T to write down the dual maximization problem with $\leq$ constraints

Solution of Minimization Problem

theorem 1 The Fundamental Principle of Duality

A minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, then the optimal value of the minimization problem is the same as the optimal value of the maximization problem.

Minimize	Maximize
$2x_1 + 5x_2 \geq 50$	$2y_1 + y_2 \leq 16$
$x_1 + 3x_2 \geq 27$	$5y_1 + 3y_2 \leq 45$
$16x_1 + 45x_2 = C$	$50y_1 + 27y_2 = P$

The minimum value of C is the same as the maximum value for P.

BUT the values for y_1, y_2 at which the maximum for P occurs are NOT the same as the x_1, x_2 values at which the minimum for C occurs!

Original Problem

$$2x_1 + 5x_2 \geq 50$$
$$x_1 + 3x_2 \geq 27$$
$$16x_1 + 45x_2 = C$$

Corner Points

(x_1,x_2)	C=
(0,10)	450
(15,4)	420
(27,0)	432

Min of C=420
at (15,4)

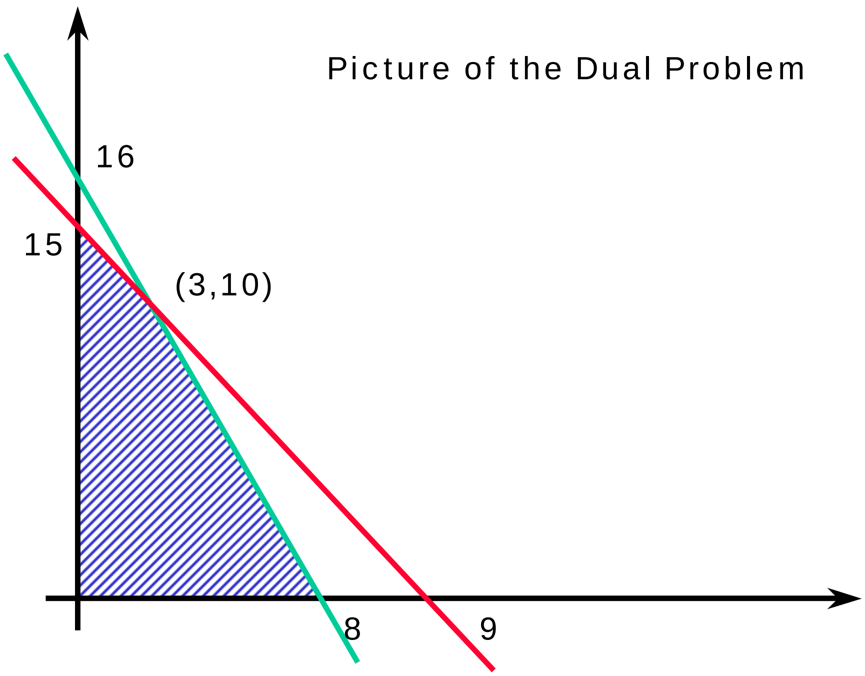
Dual Problem

$$2y_1 + y_2 \leq 16$$
$$5y_1 + 3y_2 \leq 45$$
$$50y_1 + 27y_2 = P$$

Corner Points

(y_1,y_2)	P=
(0,0)	0
(0,15)	405
(3,10)	420
(8,0)	400

Max of P=420
at (3,10)



Picture of the Dual Problem

$$\begin{array}{l} 2y_1+\ y_2+x_1 \qquad\qquad = 16 \\ 5y_1+3y_2 \quad +x_2 \quad = 45 \\ -50y_1-27y_2 \qquad +P = \ \ 0 \end{array}$$

Use x_1,x_2 as slack variables to solve the dual problem!

$$\left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 16 \\ 5 & 3 & 0 & 1 & 0 & 45 \\ \hline -50 & -27 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0.5 & 0.5 & 0 & 0 & 8 \\ 0 & 0.5 & -2.5 & 1 & 0 & 5 \\ \hline 0 & -2 & 25 & 0 & 1 & 400 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & 0 & 3 \\ 0 & 1 & -5 & 2 & 0 & 10 \\ \hline 0 & 0 & 15 & 4 & 1 & 420 \end{array}\right]$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \ y_1 & \ y_2 & \ x_1 & \ x_2 & \ P \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 3 & -1 & 0 & 3 \\ 0 & 1 & -5 & 2 & 0 & 10 \\ \hline 0 & 0 & \boxed{15} & \boxed{4} & 1 & 420 \end{array}\right] \end{array}$$

No negative indicators are left, so we found the optimal solution: (from tableau)
$y_1=3$, $y_2=10$,$x_1=0$,$x_2=0$,$P=420$

But the tableau gives more information:

$x_1=15$, $x_2=4$, $C=420$
Minimum cost of $420 when we produce
$x_1=15$, $x_2=4$ items.

The optimal solution to a minimization problem can always be found from the bottom row of the final simplex tableau for the dual problem!

Solution of a minimization problem:

Given a minimization problem with nonnegative coefficients in the objective function.

Step 1. Write all constraints as $\geq$ inequalities

Step 2. Form the dual problem.

Step 3. Use variables from minimization problem as slack variables.

Step 4. Use simplex method to solve this problem.

Step 5. If solution exist, read it from bottom row. If dual problem has no solution, neither does the original problem.

#48 p 338

A feed company stores grain in Ames and Bedford. Each month grain is shipped to Columbia and Danville for processing. The supply (in tons) in each storage and the demand (in tons) in each processing location as well as transportation cost (in $ per ton) are given below. Find shipping schedule that minimizes cost and minimum cost.

	Shipping	**Cost**	
	Columbia	**Danville**	**Supply**
Ames	$22	$38	700
Bedford	$46	$24	500
Demand	**400**	**600**	

Need shipping schedule:
Let x_1 be tons shipped from A to C
Let x_2 be tons shipped from A to D
Let x_3 be tons shipped from B to C
Let x_4 be tons shipped from B to D

$x_1+x_2 \leq 700 \qquad -x_1 - x_2 \geq -700$
$x_3+x_4 \leq 500 \qquad -x_3 - x_4 \geq -500$
$x_1+x_3 \geq 400 \qquad x_1+x_3 \geq 400$
$x_2+x_4 \geq 600 \qquad x_2+x_4 \geq 600$

$C = 22x_1+38x_2+46x_3+24x_4$
$x_1, x_2, x_3, x_4 \geq 0$

$$A = \left[\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -700 \\ 0 & 0 & -1 & -1 & -500 \\ 1 & 0 & 1 & 0 & 400 \\ 0 & 1 & 0 & 1 & 600 \\ \hline 22 & 38 & 46 & 24 & 1 \end{array}\right]$$

$$A^T = \left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 22 \\ -1 & 0 & 0 & 1 & 38 \\ 0 & -1 & 1 & 0 & 46 \\ 0 & -1 & 0 & 1 & 24 \\ \hline -700 & -500 & 400 & 600 & 1 \end{array}\right]$$

Maximize: $P=-700y_1-500y_2+400y_3+600y_4$

Subject to:
$-y_1 + y_3 \leq 22$
$-y_1 + y_4 \leq 38$
$-y_2 + y_3 \leq 46$
$-y_2 + y_4 \leq 24$ $\quad y_1, y_2, y_3, y_4 \geq 0$

$700y_1+500y_2-400y_3-600y_4+P = 0$

Introduce x_1, x_2, x_3, x_4 as slack variables.

$$\begin{array}{c|ccccccccc|c} & y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P & \\ y_1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 22 \\ y_2 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 38 \\ y_3 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 46 \\ y_4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 24 \\ \hline & 700 & 500 & -400 & -600 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$$\left[\begin{array}{rrrrrrrrr|r} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 22 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 14 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 46 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 24 \\ \hline 700 & -100 & -400 & 0 & 0 & 0 & 0 & 600 & 1 & 14400 \end{array}\right]$$

$$\left[\begin{array}{rrrrrrrrr|r} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 22 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 14 \\ 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 24 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 24 \\ \hline 300 & -100 & 0 & 0 & 400 & 0 & 0 & 600 & 1 & 23200 \end{array}\right]$$

$$\left[\begin{array}{rrrrrrrrr|r} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 22 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 14 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 38 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 38 \\ \hline 200 & 0 & 0 & 0 & 400 & 100 & 0 & 500 & 1 & 24600 \end{array}\right]$$

$y_1=0, y_2=14, y_3=22, y_4=38$

$x_1=400, x_2=100, x_3=0, x_4=500$, $P=C=24600$

5.6 Mixed Problems
The Big *M* Method

- Introduction to the Big *M* Method
- The Big *M* Method
- Minimization with Big *M*
- Summary of Solution Methods
- Larger Problems

Introduction to the Big *M* Method

Consider the problem
Maximize $P=2x_1+x_2$
Subject to $x_1+x_2 \le 10$
$-x_1+x_2 \ge 2$
$x_1, x_2 \ge 0$

$$\begin{aligned} x_1+x_2+s_1 \quad &= 10 \\ -x_1+x_2 \quad -s_2 \quad &= 2 \\ -2x_1- x_2 \quad +P &= 0 \\ x_1, x_2, s_1, s_2 &\ge 0 \end{aligned}$$

s_1 is a slack
and s_2 a surplus
variable

Non basic variables x_1,x_2
Basic variables s_1,s_2,P

Basic solution: $x_1=0$, $x_2=0$, $s_1=10$, $s_2=-2$, $P=0$
This is NOT feasible since $s_2<0$.

We can NOT just write a Simplex Tableau, we need a new variable called

ARTIFICIAL VARIABLE

This variable has no physical meaning!

For each equation that has a surplus variable we need an artificial variable:

$$-x_1+x_2-s_2+a_1=10$$

For each artificial variable, we need to introduce a PENALTY on P for using this variable:

$$P=2x_1+x_2-Ma_1$$

where M is a “large” positive number, so for each unit increase in a_1 we loose M from P.

Modified Problem

$x_1+x_2+s_1$	$= 10$	s_1 slack,
$-x_1+x_2 \quad -s_2 + \ a_1$	$= \ 2$	s_2 surplus
$-2x_1- x_2 \qquad +Ma_1 +P$	$= \ 0$	a_1 artificial
$x_1,\ x_2,\ s_1,\ s_2,\ a_1$	$\underline{>} \ 0$	M Big *M*

Preliminary Simplex Tableau

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & a_1 & P & \\
1 & 1 & 1 & 0 & 0 & 0 & 10 \\
-1 & 1 & 0 & -1 & 1 & 0 & 2 \\
\hline
-2 & -1 & 0 & 0 & M & 1 & 0
\end{array}$$

When is the preliminary tableau an initial Simplex tableau?

1. We need to be able to select enough basic variables (only one nonzero entry in column which is not in the same row as one of another basic variable)

2. The basic solution found by setting nonbasic variables equal to zero is feasible.

s_1, s_2, and P are basic (condition 1 is met), but we saw that the basic solution is not feasible.

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & P \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 10 \\
-1 & 1 & 0 & -1 & 1 & 0 & 2 \\
\hline
-2 & -1 & 0 & 0 & M & 1 & 0
\end{array}\right]
\end{array}$$

Note the -1 in the s_2 column, but the 1 in a_1
Use row operations to make a_1 basic.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -2 & -1 & 0 & 0 & M & 1 & 0 \end{array} \qquad (-M)R_2+R_3 \rightarrow R_3$$

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}$$

This is an initial Simplex tableau

$$\left[\begin{array}{cccccc|c} 2 & 0 & 1 & 1 & -1 & 0 & 8 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -3 & 0 & 0 & -1 & M+1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -3 & 0 & 0 & -1 & M+1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 6 \\ \hline 0 & 0 & \frac{3}{2} & \frac{1}{2} & M-\frac{1}{2} & 1 & 14 \end{array}\right]$$

Since M is large positive we are done.
$x_1=4, x_2=6, s_1=0, s_2=0, P=14$

Big M, slack, surplus, artificial variables

Step 1: If constraints have negative constants on the right then multiply by -1

Step 2: Introduce one slack variable for each $\leq$ constraint

Step 3: Introduce one surplus and one artificial variable for each $\geq$ constraint

Step 4: Introduce an artificial variable in each = constraint

Step 5: For each artificial variable a_i, add $-Ma_i$ to the objective function. Use the same M for each.

Big M -- Solving the problem

Step 1: Form preliminary simplex tableau

Step 2: Using row operations, eliminate M from bottom row in columns of artificial variables. This gives the preliminary Simplex tableau.

Step 3: Solve the modified problem

(A) If the modified problem has NO solution then the original has NO solution

(B) If all artificial variables are zero in the solution of the modified problem, then delete artificial variables to find the solutions of the original problem

(C) If any artificial variables are nonzero in the solution of the modified problem then the original problem has NO solution.

Minimization using Big M

Minimize $C=30x_1+30x_2+10x_3$
Subject to $x_1+x_2+x_3 \geq 6$
$2x_1+x_2+2x_3 \leq 10$
$x_1,x_2,x_3 \geq 0$

Instead maximize $P=-C=-30x_1-30x_2-10x_3$
Subject to $x_1+x_2+x_3 \geq 6$
$2x_1+x_2+2x_3 \leq 10$
$x_1,x_2,x_3 \geq 0$
Min C= -Max P

$$\begin{aligned} x_1+ \quad x_2+ \quad x_3-s_1 \quad +a_1 \qquad &= 6 \\ 2x_1+ \quad x_2+ \ 2x_3 \qquad\qquad +s_2 \quad &=10 \\ 30x_1+30x_2+10x_3 \qquad +Ma_1 \quad +P &= 0 \end{aligned}$$

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 6 \\ 2 & 1 & 2 & 0 & 0 & 1 & 0 & 10 \\ \hline 30 & 30 & 10 & 0 & M & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 1 & -1 & 1 & 0 & 0 & 6 \\ 2 & 1 & 2 & 0 & 0 & 1 & 0 & 10 \\ \hline -M+30 & -M+30 & -M+10 & M & 0 & 0 & 1 & -6M \end{array}\right]$$

After some work we get the final tableau

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 0 & -2 & 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 & -1 & 1 & 0 & 4 \\ \hline 20 & 0 & 0 & 50 & M-50 & 20 & 1 & -100 \end{array}\right]$$

$x_1=0,x_2=2,x_3=4,s_1=0,a_1=0,s_2=0,P=-100$

Min C= -Max P=-(-100)=100
at $x_1=0,x_2=2,x_3=4$

SUMMARY

Type	**Constraints**	**Right side constants**	**coeff. of objective function**	**Solution method**
Max	≤	nonnegative	any	Simplex + slack
Min	≥	any	nonnegative	dual + above
Max	Mixed (≤,≥,=)	nonnegative	any	modified with slack, surplus, artificial
Min	Mixed (≤,≥,=)	nonnegative	any	Max negative of objective